7. Classical Design in Frequency Domain

- Explores simple design techniques using Bode diagrams.
- The simple lead-lag or PID type of controllers are focused.
- Considers practical saturation and sensor noise rejection.
- Explores performance limitations of a closed-loop system due to the right half plane poles, zeros, and bandwidth constraints using the classical Bode integral relations.
### Phase-Lag Controller

A first order phase-lag controller in the following form:

\[
C(s) = \frac{K_c(s/b + 1)}{s/a + 1}, b > a.
\]

\[\Rightarrow C(0) = K_c, \quad C(\infty) = K_c \frac{a}{b}\]

A phase-lag controller is a gain compensator.

**FIGURE 7.1:** Bode diagram of a phase-lag controller \(C(s) = \frac{s/b + 1}{s/a + 1}\).
Phase-Lag Controller Design

Step 1: Find $K_c$ so that the DC gain requirements of the open-loop System $L(s) = C(s)P(s)$ are satisfied.

Step 2: Determine the desired crossover frequency $\omega_c$ and PM $\phi_c$.

Step 3: Plot the Bode diagram of $K_cP(s)$.

**FIGURE 7.2:** Phase-lag controller design.
Phase-Lag Controller Design (Cont.)

Step 4: Check the phase plot of $K_c P(s)$ at the desired crossover frequency $\omega_c$. If $180^\circ + \angle K_c P(j \omega_c) \geq \phi_c + 5^\circ$, then a phase-lag controller can be designed to satisfy the design specifications. (We expect $C(s)$ to contribute about $-5^\circ$ at $\omega_c$.)

Step 5: Choose,
\[ b \approx 0.1\omega_c, \quad a = \frac{b}{K_c|P(j\omega_c)|}. \]

Step 6: Finally, a phase-lag controller is given by
\[ C(s) = \frac{K_c(s/b + 1)}{s/a + 1}. \]
Example

A unity feedback system

\[ P(s) = \frac{8(s + 10)}{s(s + 3)(s + 20)}. \]

Specification
- The step response of the system has no more than 20% maximum overshoot and 8 [sec] of maximum settling time (2% tolerance)
- The steady-state error with respect to a ramp input is no greater than 0.05

\[ \text{PO} \leq 20\% \text{ is equivalent to a damping } \zeta \geq 0.45 \text{ or PM } \geq 47^\circ \]
Example (Cont.)

The settling time

\[ t_s = \frac{4}{\zeta \omega_n} \leq 8 \]

gives \( \omega_n \geq 0.5 / \zeta = 1.11 \).

We need
\( \omega_c \geq 0.8 \omega_n = 0.89 \) when \( \zeta \approx 0.45 \).

Finally, \( \frac{1}{K_v} \leq 0.05 \) gives \( K_v \geq 20 \).

Since
\[ K_v = \lim_{s \to 0} sC(s)P(s) = 4C(0)/3, \]

We need \( C(0) \geq 15 \).
Example (Cont.)

Now take $K_c = 15$.
From the Bode diagram, we can choose $\omega_c = 2$ and a lag controller can be designed to satisfy the PM and the $K_v$ condition since $180^\circ + \angle P(j\omega) = 62^\circ > 47^\circ + 5^\circ$.

Pick $b = 0.1\omega_c = 0.2$. Then

$$a = \frac{b}{K_c|P(j\omega)|} = \frac{0.2}{8.4432} = 0.0237$$

Which gives a phase-lag controller

$$C(s) = \frac{K_c(s/b + 1)}{s/a + 1} = \frac{15(s/0.2 + 1)}{s/0.0237 + 1} = \frac{15(5s + 1)}{42s + 1}.$$ 

The compensated loop transfer function has $56.75^\circ$ PM.
Example (Cont.)

The overshoot is 16% and the settling time is about 6.94 [sec].

A controller design process is an iterative process and usually involves trial-and-error steps. If one chooses $\omega_c = 3$, then the controller is

$$C_1(s) = \frac{50s + 15}{16.23s + 1}$$

It will result in a 23% overshoot and the specification will not be satisfied.
A PI controller in the following form:

\[ C(s) = \frac{K_c(s/b + 1)}{s}. \]

\[ \Rightarrow C(0) = \infty, \quad C(\infty) = \frac{K_c}{b}. \]

A PI controller increases the system type by 1.
PI Controller Design

Step 1: Determine the desired crossover frequency $\omega_c$ and PM $\phi_c$.

Step 2: Plot the Bode diagram of $KP(s)$ for any given $K$ (e.g., $K = 1$).

Step 3: Check the phase plot of $KP(s)$ at the $\omega_c$. If $180^\circ + \angle KP(j\omega_c) \geq \phi_c + 5^\circ$, then a PI controller be designed to satisfy the design specifications.

Step 4: Choose $b \approx 0.1\omega_c$, $K_c = \frac{b}{|P(j\omega_c)|}$.

Step 5: Finally, a PI controller is given by

$$C(s) = \frac{K_c(s/b + 1)}{s}.$$
Example

A plant transfer function is given by

\[ P(s) = \frac{30}{(s + 5)(s^2 + 4s + 8)} \]

Specification
- A PM of at least $40^\circ$ and a crossover frequency of at least 2.
- The steady-state error with respect to a step input is no more than 0.01.

The phase of $P(s)$ at $\omega = 2$ is about $-85^\circ$.
A phase-lag or PI controller can be design to satisfy the design specifications.
Example (Cont.)

We choose $\omega_c = 3$ for fast response. Then

$$\angle P(j\omega) \approx -126^\circ \text{ and } |P(j\omega)| = 0.4273.$$ 

Take $b = 0.1\omega_c = 0.3$ and

$$K_c = \frac{b}{|P(j\omega_c)|} = \frac{0.3}{0.4273} = 0.7021.$$ 

We then have a PI controller

$$C(s) = \frac{K_c(s/b + 1)}{s} = \frac{0.7021(s/0.3 + 1)}{s}.$$ 

The compensated system is type 1 and has a PM of $48^\circ$ with $\omega_c = 3$. All design specifications are met.

A Phase-lag controller can be also designed.

$$C(s) = \frac{133.3(s/0.3 + 1)}{s/0.0053 + 1}$$
Phase-Lead Controller

A first-order phase-lead controller in the following form:

\[ C(s) = \frac{K_c(s/b + 1)}{s/a + 1}, \quad b < a. \]

A phase-lead controller is a phase compensator.

**FIGURE 7.7:** Phase-lead controller \( C(s) = \frac{s/b + 1}{s/a + 1} \) \((b < a)\).
Phase-Lead Controller

Note that the maximum phase of this compensator is given by

\[
\phi_{\text{max}} = \sin^{-1} \frac{a - b}{a + b} = \sin^{-1} \frac{a/b - 1}{a/b + 1} \quad \text{or} \quad \frac{a}{b} = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}},
\]

The frequency where the maximum is achieved as \( \omega_{\text{max}} = \sqrt{ba} \).

**FIGURE 7.8:** Maximum phase angle for \( C(s) = \frac{s/b + 1}{s/a + 1} \): \( \phi_{\text{max}} = \sin^{-1} \frac{a - b}{a + b} \).
Phase-Lead Controller Design

Step 1: Find $\omega_c$ so that the DC gain requirements of the open-loop system $L(s) = K_1P(s)$ are satisfied.

Step 2: Determine the desired crossover frequency $\omega_{desired}$, PM $\phi_{desired}$, and so on.

Step 3: Plot the Bode diagram of $K_1P(s)$ and calculate the crossover frequency $\omega_c$.

Step 4: If $\omega_1 \leq \omega_{desired}$, let $\omega_c = \omega_{desired}$ and let

$$\phi_{\text{max}} = \phi_{\text{desired}} - \angle K_1P(j\omega_c) - 180^\circ.$$ 

Find $a$ and $b$ from the following equations:

$$\frac{a}{b} = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}}, \quad \omega_c = \sqrt{ba}.$$ 

Find $K_c$ such that

$$\left| \frac{K_c(j\omega_c/b + 1)}{j\omega_c/a + 1} \right| |P(j\omega_c)| = 1.$$ 

If $K_c \geq K_1$, go to step 6. Otherwise, go to step 5.
Phase-Lead Controller Design

Step 5: If \( \omega_1 \approx \omega_{\text{desired}} \) or \( \omega_1 \geq \omega_{\text{desired}} \), let \( K_c = K_1 \).

Estimate the phase \( \phi_{\text{max}} \) needed by examining the Bode diagram in the frequency range \( \geq \omega_1 \).

Let

\[
\frac{a}{b} = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}}
\]

and let \( \omega_c \) be such that

\[
|K_1 P(j\omega_c)| = \sqrt{\frac{b}{a}}.
\]

Find \( b \) and \( a \) by setting

\[
\omega_c = \sqrt{ab}.
\]

Step 6: Let a lead controller be given by

\[
C(s) = \frac{K_c(s/b + 1)}{s/a + 1}.
\]

Step 7: Plot the Bode diagram of \( C(s)P(s) \) and check the design specifications. Adjust \( K_c, a, b \) if necessary.
Example 1

A plant transfer function is given by

\[ P(s) = \frac{100}{s(s + 20)}. \]

Specifications
- The system has a PM of at least $50^\circ$ and a crossover frequency of at least 40.
- The steady-state error with respect to a ramp input is no more than 0.1.
Example 1 (Cont.)

The steady-state error requirement is easily satisfied for any \( K_c \geq K_1 := 2 \). The \( \omega_1 = 10 \), which is much smaller than \( \omega_{desired} = 40 \).

Since \( \angle K_1 P(j \omega_{desired}) \approx -154^\circ \), the positive phase needed at \( \omega_{desired} = 40 \) to guarantee the PM is

\[
\phi_{max} = 50 + 154 - 180 = 24^\circ.
\]

Let

\[
\frac{a}{b} = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}} = 2.3712,
\]

\[
\sqrt{ba} = \omega_{desired} = 40,
\]

i.e.,

\[
a \approx 61.6, \quad b \approx 26.
\]
Example 1 (Cont.)

Now determine $K_c$ so that the crossover frequency is $\omega_c = \omega_{\text{desired}} = 40$, i.e.,

$$|C(j40)P(j40)| = \left| \frac{K_c(j40/26 + 1)}{j40/61.6 + 1} P(j40) \right| = 0.086 \quad K_c = 1.$$ 

We thus need $K_c = 11.6$, which is greater than $K_1$. Hence the final controller is given by

$$C(s) = \frac{11.6(s/26 + 1)}{s/61.6 + 1}$$

All design specifications are satisfied.

FIGURE 7.10: Example 7.6 – $K_1P(s)$ (dashed) and $C(s)P(s)$ (solid).
Example 2

A plant transfer function is given by

\[ P(s) = \frac{100}{s(s + 20)}. \]

Specifications

- The system has a PM of at least \(50^\circ\) and a crossover frequency of at least 40.
- The steady-state error with respect to a ramp input is no more than \(0.01\).

The steady-state error requirement is easily satisfied for any \(K_c \geq K_1 := 20\).

The \(\omega_1 = 42.5\), which is greater than the desired \(\omega_{desired} = 40\).

Thus we can take \(K_c = K_1 = 20\).
Example 2 (Cont.)

Now, examining the phase diagram of $K P(s)$, the lead controller will need to provide a phase of $30^\circ$.

Let

$$\phi_{\text{max}} = 40^\circ \quad \text{and} \quad \frac{a}{b} = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}} = 5.6254.$$  

Let $\omega_c$ be such that

$$|K P(j \omega_c)| = \sqrt{\frac{1}{5.6254}} = 0.42.$$  

From the Bode diagram, $\omega_c \approx 67$. Let

$$\sqrt{ba} = \omega_c = 67,$$

i.e.,

$$a \approx 158.9, \quad b \approx 28.25.$$  

Thus we have

$$C(s) = \frac{20(s/28.25 + 1)}{s/156.9 + 1}.$$  

The actual PM achieved is $60.7^\circ$. 
PD Controller

A PD controller in the following form:

$$C(s) = K_c(s/b + 1).$$

The phase of this compensator is given by

$$\phi = \tan^{-1} \frac{\omega}{b}.$$
PD Controller Design

Step 1: Find $K_1$ so that the DC gain requirements of the open-loop system $L(s) = K_1 P(s)$ are satisfied.

Step 2: Determine the desired crossover frequency $\omega_{\text{desired}}, \text{PM}, \phi_{\text{desired}}$, and so on.

Step 3: Plot the Bode diagram of $K_1 P(s)$ and calculate the crossover frequency $\omega_1$.

Step 4: If $\omega_1 \not\approx \omega_{\text{desired}}$, let $\omega_c = \omega_{\text{desired}}$ and let $\phi_{\text{max}} = \phi_{\text{desired}} - \angle K_1 P(j \omega_c) - 180^\circ$.

Find $b$ from the following equation

$$b = \frac{\omega_c}{\tan \phi_{\text{max}}}.$$ 

Find $K_c$ such that

$$K_c \left| j \omega_c / b + 1 \right| |P(j \omega_c)| = 1.$$ 

If $K_c \not\approx K_1$, go to step 6. Otherwise, go to step 5.
PD Controller Design (Cont.)

Step 5: If $\omega_1 \approx \omega_{desired}$, or $\omega_1 \geq \omega_{desired}$, let $K_c = K_1$. Estimate the phase $\phi_{max}$ needed by examining the Bode diagram in the frequency range $\geq \omega_1$. Let $\omega_c$ and $b$ such that $1 \geq \omega_c \geq \omega_1$. Find $b$ from the following equation

$$b = \frac{\omega_c}{\tan \phi_{max}}.$$  

and

$$|K_c(j\omega_c/b + 1)P(j\omega_c)| \approx 1.$$  

Step 6: Let a PD controller be given by

$$C(s) = K_c(s/b + 1).$$  

Step 7: Plot the Bode diagram of $C(s)P(s)$ and check the design specifications. Adjust $b$ and $K_c$ if necessary.
Example

A feedback system with a plant transfer function

\[ P(s) = \frac{50}{s^2(s + 10)}. \]

Design a PD feedback controller.

Specifications
- A PM of at least \(40^\circ\) and a crossover frequency of at least 1.

Since \(\angle P(j10) = -225^\circ\) and the maximum phase of the PD controller is \(90^\circ\), the crossover frequency must be less than 10.

FIGURE 7.13: Example 7.9 - \(P(s)\) (dashed) and \(C(s)P(s)\) (solid).
Example (Cont.)

Let $\omega_c = 1$ and note that $\angle P(j1) = -185.7^\circ$. Hence, to achieve at least $40^\circ$ PM at $\omega_c = 1$, we shall take $\phi_{\text{max}} = 50^\circ$. Then

$$b = \frac{\omega_c}{\tan \phi_{\text{max}}} = 0.8391.$$ 

Let $K_c$ be such that

$$K_c|j\omega_c/b + 1| |P(j\omega_c)| = 1.$$ 

Then $K_c = 0.1292$. Thus a PD controller is given by

$$C(s) = K_c(s/b + 1) = 0.1292(s/0.8391 + 1).$$

The loop gain greater than 20 [dB] for all frequencies $\omega \leq 1$ with the same PM. This would be satisfied for $K_1P(s)$ with $K_1 = 2.01$. Since the crossover frequency for $K_1P(s)$ is greater than 3, it is necessary to pick $\omega_c \geq 3$. 
Example (Cont.)

Let $\omega_c = 7$. We then need $\phi_{\text{max}} = 75^\circ$ and

$$b = \frac{\omega_c}{\tan \phi_{\text{max}}} = \frac{7}{3.7321} = 1.8756.$$ 

Let $K_c$ be such that

$$K_c|j\omega_c/b + 1| |P(j\omega_c)| = 1$$

i.e., $K_c = 3.096 > K_1$.

We then obtain a PD controller as

$$C_1(s) = 3.096(s/1.8756 + 1)$$

**FIGURE 7.14**: Example 7.9 – $K_1 P(s)$ (dashed) and $C_1(s) P(s)$ (solid).
A lead-lag controller can take the following general form:

\[
C(s) = \frac{K_c \left( \frac{s}{b_1} + 1 \right)}{s/a_1 + 1} \frac{s/b_2 + 1}{s/a_2 + 1}, \quad a_1 < b_1 < b_2 < a_2.
\]

**FIGURE 7.15:** Lead-lag controller \( C(s) = \frac{s/b_1 + 1}{s/a_1 + 1} \frac{s/b_2 + 1}{s/a_2 + 1} \).
Lead-Lag Controller Design

Step 1: Find $K_c$ so that the DC gain requirements of the open-loop system $L(s) = K_cP(s)$ are satisfied.

Step 2: Determine the desired crossover frequency $\omega_{c,PM} \phi_{desired}$.

Step 3: Plot the Bode diagram of $K_cP(s)$ and calculate the phase $\phi_{\max}$ needed at $\omega_c$ in order to achieve the desired PM:

$$\phi_{\max} = \phi_{desired} - \angle K_cP(j\omega_c) - 180^\circ + 5^\circ.$$

Step 4: Choose $a_2$ and $b_2$ such that

$$\frac{a_2}{b_2} = \frac{1 + \sin \phi_{\max}}{1 - \sin \phi_{\max}}, \ \omega_c = \sqrt{b_2a_2}.$$

Let

$$C_{lead}(s) = \frac{K_c(s/b_2 + 1)}{s/a_2 + 1}.$$

Step 5: Choose

$$b_1 \approx 0.1\omega_c, \ a_1 = \frac{b_1}{|C_{lead}(j\omega_c)P(j\omega_c)|}.$$

Step 6: Plot the Bode diagram of $C(s)P(s)$ and check the design specifications.
A PID controller can be designed by combining the design process for PD and PI controllers.

\[ C(s) = \frac{K_c}{s} \left( \frac{s}{b_1} + 1 \right) \left( \frac{s}{b_2} + 1 \right) \]

**FIGURE 7.16:** Bode diagram of a PID controller \( C(s) = \frac{(s/b_1 + 1)(s/b_2 + 1)}{s} \).
PID Controller Design

Step 1: Determine the desired crossover frequency $\omega_c \text{PM } \phi_{\text{desired}}$.

Step 2: Plot the Bode diagram of $P(s)$ and calculate the phase $\phi_{\text{max}}$ needed at $\omega_c$ in order to achieve the desired PM:

$$\phi_{\text{max}} = \phi_{\text{desired}} - \angle P(j\omega_c) - 180^\circ + 5^\circ.$$ 

Step 3: Choose $b_2$ such that

$$b_2 = \frac{\omega_c}{\tan \phi_{\text{max}}}.$$ 

Let

$$C_{pd}(s) = s/b_2 + 1.$$ 

Step 4: Choose

$$b_1 \approx 0.1\omega_c, \quad K_c = \frac{b_1}{|C_{pd}(j\omega_c)P(j\omega_c)|}.$$ 

Step 5: A PID controller is given by

$$C(s) = \frac{K_c(s/b_1 + 1)(s/b_2 + 1)}{s}.$$ 

Step 6: Plot the Bode diagram of $C(s)P(s)$ and check the design specifications.
Derivative Control

The simple feedback control system with a PD controller.

\[ C(s) = K_P(1 + T_D s) \]

FIGURE 7.25: Standard implementation of a PD controller.

Since a pure derivative is not physically realizable, it is usually implemented with suitable filtering.

\[ C(s) = K_P \left( 1 + \frac{T_D s}{T_S + 1} \right) \quad \text{with a small } T > 0 \]
Example

The feedback system with

\[ C(s) = 5(s + 5) = 25(0.2s + 1), \quad P(s) = \frac{10}{s(s + 5)(s + 10)}. \]

A SIMULINK diagram of this feedback system with the PD controller as

\[ C(s) = 25 \left(1 + \frac{0.2s}{Ts + 1}\right) \text{ with } T = 0.01. \]

**FIGURE 7.26:** Example 7.13 – A PD-controlled system with actuator saturation.
Example (Cont.)

The response of the feedback system without actuator saturation and with saturation limit $u_{\text{max}} = 5, 10, \text{and } 20$. The control signal due to the derivative control is very large when the command signal jumps.
An alternative is to use a rate sensor to measure the derivative of the output directly or implement the derivative control in the feedback.

**FIGURE 7.28:** PD control using an inner loop feedback with a rate sensor or derivative.
Example (Cont.)

It can reduce actuator saturation because of the abrupt change in input signals.

The system responses are much smoother.

**FIGURE 7.29:** Example 7.13 with inner loop feedback – Responses to a square wave of 0.2 [Hz] with actuator saturation $u_{\text{max}} = 5$ (dotted), $u_{\text{max}} = 10$ (dash-dot), $u_{\text{max}} = 20$ (dashed), and $u_{\text{max}} = \infty$ (solid).
Example (Cont.)

Without the filtering, much more noisy than the control signal with a suitable filtering.

\[
\frac{1}{Ts + 1} = \frac{1}{0.01s + 1}
\]

It filters out the high-frequency noise \( n(t) = 0.01 \sin 1000t \).

**FIGURE 7.30:** Example 7.13 – PD control using an inner-loop derivative feedback with a sensor noise \( n(t) = 0.01 \sin 1000t \) and without actuator saturation. Top: output \( y(t) \) for \( T = 0.01 \) (solid) and \( T = 0 \) (dashed); Middle: actuator output \( u(t) \) for \( T = 0 \); Bottom: actuator output \( u(t) \) for \( T = 0.01 \).
Example (Cont.)

We assume that the actuator has a saturation limit $u_{\text{max}} = 20$, the system responses are quite different.

Both the outputs and the actuator signals with $T = 0.01$ are better than $T = 0$, with actuator saturation.

![Graphs showing system responses and actuator signals.]
Alternative PID Implementation

When there is a jump in command signal, the output of the proportional controller may also generate a large control signal, which may saturate the actuator.

**FIGURE 7.32**: Standard implementation of a PID controller.

**FIGURE 7.33**: PID control using an inner-loop feedback with a rate sensor/output derivative.
Alternative PID Implementation

Assume that the PID controller is simply a PD controller:

\[
\text{PID} = K_P (1 + T_D s). \quad F(s) = 1 \quad P(s) = \frac{b(s)}{a(s)}.
\]

The open-loop transfer functions of the standard and alternative implementation are respectively.

\[
L_s(s) = \frac{K_P (1 + T_D s)b(s)}{a(s)}, \quad L_a(s) = \frac{K_P P(s)}{1 + K_P T_D s P(s)} = \frac{K_P b(s)}{a(s) + K_P T_D s b(s)}.
\]

The standard implementation may have a shorter rise time but a larger overshoot compared to the alternative implementation.

\[
K_v = \frac{K_P b(0)}{a_1(0)}, \quad \text{standard implementation}
\]

\[
K_v = \frac{K_P b(0)}{a_1(0) + K_P T_D b(0)} \quad \text{alternative implementation}.
\]

The alternative implementation may result in a larger steady-state error compared to the standard implementation.
Bode’s Gain and Phase Relation

Bode’s gain-phase integral relation has been used as an important tool to express design constraints.

Theorem 1. Let $L(s)$ be a stable and minimum phase transfer function.

$$\angle L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \ln |L(j\omega)|}{d\nu} \ln \coth \frac{\nu}{2} d\nu$$

(7.1)

where $\nu := \ln(\omega/\omega_0)$.

$$\frac{1}{\pi} \int_{-\alpha}^{\alpha} \ln \coth \frac{\nu}{2} d\nu = \begin{cases} 
1.1406 \text{ (rad)}, & \alpha = \ln 3 \\
1.3146 \text{ (rad)}, & \alpha = \ln 5 \\
1.443 \text{ (rad)}, & \alpha = \ln 10 \\
\end{cases}$$

$$= \begin{cases} 
65.3^\circ, & \alpha = \ln 3 \\
75.3^\circ, & \alpha = \ln 5 \\
82.7^\circ, & \alpha = \ln 10. \\
\end{cases}$$

**FIGURE 7.42:** The function $\ln \coth \frac{\nu}{2}$ vs. $\nu$. 

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Bode’s Gain and Phase Relation

$\angle L(j\omega_0)$ will be large if the gain $L(j\omega)$ attenuates slowly near $\omega_0$, and small if the gain $L(j\omega)$ attenuates rapidly near $\omega_0$.

If $\frac{d \ln |L(j\omega)|}{d\omega} = -\ell$, then

$\angle L(j\omega_0) < \left\{ \begin{array}{ll}
-\ell \times 65.3^\circ, & \text{if the slope of } L(j\omega) = -\ell \text{ for } \frac{1}{3} \leq \frac{\omega}{\omega_0} \leq 3 \\
-\ell \times 75.3^\circ, & \text{if the slope of } L(j\omega) = -\ell \text{ for } \frac{1}{5} \leq \frac{\omega}{\omega_0} \leq 5 \\
-\ell \times 82.7^\circ, & \text{if the slope of } L(j\omega) = -\ell \text{ for } \frac{1}{10} \leq \frac{\omega}{\omega_0} \leq 10.
\end{array} \right.$

The behavior of $\angle L(j\omega)$ is important near the crossover frequency $\omega_c$, where $|L(j\omega_c)| = 1$ since $\pi + \angle L(j\omega)$ is the PM of the feedback system.

Thus, it is important to keep the slope of $L(j\omega)$ near $\omega_c$ not much smaller than -1 for a reasonably wide range of frequencies in order to guarantee some reasonable performance.
Bode’s Gain and Phase Relation

Theorem 2. Let \( z_1, z_2, \ldots, z_k \) be the right half plane zeros of a stable \( L(s) \). Then

\[
\angle L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\ln |L(j\omega)|}{d\nu} \ln \coth \frac{\nu}{2} d\nu + \sum_{i=1}^{k} \angle \frac{-j\omega_0 + z_i}{j\omega_0 + z_i}. \tag{7.2}
\]

Proof. \( L(s) \) can be factorized as

\[ L(s) = \frac{-s + z_1}{s + z_1} \frac{-s + z_2}{s + z_2} \cdots \frac{-s + z_k}{s + z_k} L_{mp}(s), \]

where \( L_{mp}(s) \) is the stable and minimum phase and \( |L(j\omega)| = |L_{mp}(j\omega)| \). Hence

\[
\angle L(j\omega_0) = \angle L_{mp}(j\omega_0) + \prod_{i=1}^{k} \frac{-j\omega_0 + z_i}{j\omega_0 + z_i}
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\ln |L_{mp}(j\omega)|}{d\nu} \ln \coth \frac{\nu}{2} d\nu + \sum_{i=1}^{k} \angle \frac{-j\omega_0 + z_i}{j\omega_0 + z_i},
\]

Which gives

\[
\angle L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\ln |L(j\omega)|}{d\nu} \ln \coth \frac{\nu}{2} d\nu + \sum_{i=1}^{k} \angle \frac{-j\omega_0 + z_i}{j\omega_0 + z_i}.
\]
Bode’s Gain and Phase Relation

Since the slope of $|L(s)|$ near the crossover frequency is no greater than -1, meaning that the phase due to the minimum phase part $L_{mp}(s)$ of $L(s)$ will be no greater than $-90^\circ$, the crossover frequency must satisfy

$$\omega_c < \frac{\omega}{2} \quad (7.3)$$

in order to guarantee closed-loop stability and performance.

**FIGURE 7.43:** Phase $\phi_1(\omega_0/z)$ due to a real zero $z > 0$. 
Bode’s Gain and Phase Relation

Next, suppose \(|L(s)|\) has a pair of complex right half plane zeros at \(z = x \pm jy\) with \(x > 0\); then

\[
\phi_2(\omega_0/|z|) := \angle \frac{-j\omega_0 + z}{j\omega_0 + z} \frac{-j\omega_0 + \bar{z}}{j\omega_0 + \bar{z}} \bigg|_{\omega_0=|z|/|z|/2,|z|/3,|z|/4}
\]

\[
\approx \begin{cases} 
-180^\circ, & -106.26^\circ, & -73.7^\circ, & -56^\circ, & \text{Re}(z) \gg \text{Im}(z) \\
-180^\circ, & -86.7^\circ, & -55.9^\circ, & -41.3^\circ, & \text{Re}(z) \approx \text{Im}(z) \\
-360^\circ, & 0^\circ, & 0^\circ, & 0^\circ, & \text{Re}(z) \ll \text{Im}(z)
\end{cases}
\]

The crossover frequency must satisfy

\[
\omega_c < \begin{cases} 
|z|/4, & \text{Re}(z) \gg \text{Im}(z) \\
|z|/3, & \text{Re}(z) \approx \text{Im}(z) \\
|z|, & \text{Re}(z) \ll \text{Im}(z)
\end{cases} \quad (7.4)
\]

in order to guarantee closed-loop stability and performance.
Let

\[ L(s) = \frac{K(10-s)}{(s+10)(s+2)(s+3)} = \frac{10-s}{s+10} \frac{K}{(s+2)(s+3)}. \]

Then

\[ L_m(s) = \frac{K}{(s+2)(s+3)}. \]

The stability margin of the system with \( L(s) \) is very small when the \( \omega_c \) is half of \( z = 10 \).

The nonminimum phase \( L(s) \) in the loop becomes unstable when \( \omega_c = z / 2 = 5 \).
Bode’s Sensitivity Integral

Theorem 3. Let $L(s)$ be the open-loop transfer function with at least two more poles than zeros and let $p_1, p_2, \ldots, p_m$ be the open right half plane poles of $L(s)$. The following Bode’s sensitivity integral then holds:

$$\int_{0}^{\infty} \ln |S(j\omega)| d\omega = \pi \sum_{i=1}^{m} p_i. \quad (7.5)$$

If $L(s)$ is stable, then the integral simplifies to

$$\int_{0}^{\infty} \ln |S(j\omega)| d\omega = 0. \quad (7.6)$$
Water bed effect
There will exist a frequency range over which the magnitude of the sensitivity function exceeds 1 if it is to be kept below 1 at other frequencies.

We must have $|S(j\omega)| \approx 1, \forall \omega \in [\omega_h, \infty)$ for some high-frequency $\omega_h$. 

FIGURE 7.16: Gunter Stein’s interpretation of the water bed effect: Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.
Example

The feedback system is designed such that

$$|S(j\omega)| \leq \epsilon < 1, \quad \forall \omega \in [0, \omega_l]$$

where $\epsilon > 0$ is a given constant.

Also suppose that

$$|L(j\omega)| \leq \frac{M_h}{\omega^2} \leq \tilde{\epsilon} < 1, \quad \forall \omega \in [\omega_h, \infty)$$

where $\omega_h > \omega_l$, and $M_h > 0$ is a given constant. Then for $\omega \geq \omega_h$,

$$|S(j\omega)| \leq \frac{1}{1 - |L(j\omega)|} \leq \frac{1}{1 - \frac{M_h}{\omega^2}}$$